Linear Differential Equations With Variable Coefficients

Introduction →

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = Q.$$

where $P_0, P_1, P_2, \ldots P_n$ & Q are functions of x, is called a linear differential equation with variable coefficients.

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Cauchy's Linear Equation →

A linear equation of the form

$$P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n x^0 y = Q(x)$$

where $P_0, P_1, P_2, \ldots P_n$ are real const. & $Q(x)$ is a function of x, is called Cauchy's linear equation.

Working Rule to solve Cauchy's Linear Equation

Step-I → Put $x = e^z$

i.e $z = \log x$, $x > 0$

Step-II → Put $\frac{d}{dz} = \theta$ so that

$xD = \theta$, $x^2D^2 = \theta(\theta-1)$, $x^3D^3 = \theta(\theta-1)(\theta-2)$

$\dots \quad x^nD^n = \theta(\theta-1)(\theta-2)\dots(\theta-\overline{n-1})$

Step-III → Putting these in the given Equation, we get

$$\left[P_0\,\theta(\theta-1)\dots(\theta-\overline{n-1}) + P_1\,\theta(\theta-1)\dots(\theta-\overline{n-2}) + \dots + P_n\right]y = Q(e^z)$$

which is a linear Equation with const. coefficients and solve for y in terms of z.

Step-IV → Put $z = \log x$ to get the req. sol.

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ILLUSTRATIVE EXAMPLES

Example 1 → Solve the differential Equation

$$x^3\frac{d^3y}{dx^3}+6x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}-4y=0$$

Sol. → The given differential Equation is

$$x^3\frac{d^3y}{dx^3}+6x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}-4y=0 \quad \text{—— (1)}$$

or in S.F $\left(x^3D^3+6x^2D^2+4xD-4\right)y=0$ —(2)

Put $x=e^z$ or $\log x=z$, $x>0$.

and $Dx=\theta$, $x^2D^2=\theta(\theta-1)$, $x^3D^3=\theta(\theta-1)(\theta-2)$ in (2), we get.

$\therefore\ [\theta(\theta-1)(\theta-2)+6\theta(\theta-1)+4\theta-4]y=0$

or $[\theta^3-3\theta^2+2\theta+6\theta^2-6\theta+4\theta-4]y=0$

or $(\theta^3+3\theta^2-4)y=0$

$\therefore$ A.E is $\theta^3+3\theta^2-4=0$ Here $\theta=1$ is one root of Eq.

or $(\theta-1)(\theta^2+4\theta+4)=0$

or $(\theta-1)(\theta+2)^2=0$

$\boxed{\text{or } \theta=1,-2,-2}$

$\therefore$ general sol. is

$y=c_1e^z+e^{-2z}(c_2+c_3z)$

or $\boxed{y=c_1x+x^{-2}(c_2+c_3\log x)}$ Ans

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Example 2 → Solve

$$(x^2D^2 + 2xD - 2)y = 0$$

Sol:- The given differential Equation is

$$(x^2D^2 + 2xD - 2)y = 0 \quad \text{—(1)}$$

Put $x = e^z$ i.e $z = \log x$, $x > 0$.

$\&$ $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$ in (1)

$\therefore$ $[\theta(\theta-1) + 2\theta - 2]y = 0$

or $(\theta^2 - \theta + 2\theta - 2)y = 0$.

$\therefore$ A.E is $\theta^2 + \theta - 2 = 0$

or $(\theta - 1)(\theta + 2) = 0$

$\therefore$ $\boxed{\theta = 1, -2}$

$\therefore$ general sol is

$y = c_1 e^z + c_2 e^{-2z}$

$\boxed{\text{or } y = c_1 x + c_2 x^{-2}}$

Example-3 → Solve $x^2\frac{d^2y}{dx^2} + 9x\frac{dy}{dx} + 25y = 50$

Sol:) The given differential Equation is

$$x^2D^2y + 9xDy + 25y = 50 \quad \text{—(1)}$$

or $[x^2D^2 + 9xD + 25]y = 50$

Put $x = e^z$ i.e $z = \log x,\ x>0$

& $xD = \theta$; $x^2D^2 = \theta(\theta-1)$ in (1)

$\therefore [\theta(\theta-1) + 9\theta + 25]y = 50$

$\therefore$ A.E is $\theta^2 - \theta + 9\theta + 25 = 0$

or $\theta^2 + 8\theta + 25 = 0$

or $\boxed{\theta = -4 \pm 3i}$

$\therefore$ C.F $= e^{-4x}[c_1\cos 3z + c_2\sin 3z]$

$= x^{-4}[c_1\cos(3\log x) + c_2\sin(3\log x)]$

Now, P.I $= \frac{1}{f(\theta)}.\theta. = \frac{1}{\theta^2+8\theta+25}(50)$

$= 50.\frac{1}{\theta^2+8\theta+25}e^{0z}$

$= 50.\frac{1}{0+0+25}e^{0z} = 50.\frac{1}{25}e^{0z} = 2.$

$\therefore$ Complete sol. is

$y = 2 + x^{-4}[c_1\cos(3\log x) + c_2\sin(3\log x)]$